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Classification of contact structures associated with the CR-structure of the complex indicatrix

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Abstract

By regarding the complex indicatrix as an embedded CR-hypersurface of the holomorphic tangent bundle in a fixed point, we analyze some aspects of the relations between its CR structure and the considered contact structure. Moreover, using the classification of the almost contact metric structures associated with a strongly pseudo-convex CR-structure, of D. Chinea and C. Gonzales, we determine the classes corresponding to the natural contact structure of the complex indicatrix and the new structures obtained under a gauge transformation.

1 Introduction

Many geometers dedicated their attention to the study of relationships between the geometric properties of a Riemannian or a Finsler manifold M and those of its unit tangent sphere bundle, named also indicatrix ([3, 4, 5, 7, 10, 14], etc.). This research field, extended to the complex Finsler spaces, is very interesting and important, mainly because the complex indicatrix is a compact and strictly convex set surrounding the origin, used in the study of the volume of Finsler manifolds or in the study of the Laplacian or Hodge theories.

However, the main purpose of the present paper is to present the indicatrix of a complex Finsler space from an algebraic point of view by finding the

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classes corresponding to the natural contact structure of the complex indicatrix and the new contact structure obtained under a gauge transformation. In this sense, we used the complete classification for almost contact metric manifolds obtained by D. Chinea and C. Gonzales [6] through the study of the covariant derivative of the fundamental 2-form. This represents an algebraic decomposition of the geometric structures and for every invariant subspace is assigned a different class of contact manifolds.

Firstly, in Section 1, we make a short overview of the concepts and terminology specific to complex Finsler manifold M, as in [1, 12]. By taking $z \in M$ an arbitrary point, the punctured holomorphic tangent bundle T'_zM can be locally viewed as a Kähler manifold and the complex indicatrix is a real hypersurface, i.e. a CR hypersurface of T'_zM. Thus, the CR structure of the complex indicatrix and its associated natural contact structure are studied in Section 2. Using these and the classification of almost contact metric structures associated with a strongly pseudo-convex CR-structure, developed by P. Matzeu and M.I. Munteanu [11] in the light of the 12 mutual classes introduced by D. Chinea and C. Gonzales [6], we determine in Section 3 the classes corresponding to the natural contact structure of the complex indicatrix. Moreover, there are analyzed the classes of different types of almost contact metric structures associated with the same CR-structure of the complex indicatrix, introduced under a gauge transformation.

Let M be an n-dimensional complex manifold, with $z := (z^k),\ k = 1,..,n$, complex coordinates on a local chart. The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. $T'M$ is in its turn a $2n$-dimensional complex manifold, of local coordinates in a local chart in $u \in T'M$ given as $u := (z^k, \eta^k),\ k = 1,..,n$.

Definition 1.1. A *complex Finsler space* is a pair (M, F), with $F : T'M \to \mathbb{R}^+,\ F = F(z, \eta)$ a continuous function that satisfies the following conditions:

i. F is a smooth function on $\widetilde{T'M} := T'M \setminus \{0\}$;

ii. $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii. $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, $\forall \lambda \in \mathbb{C}$;

iv. the Hermitian matrix $\left(g_{i\bar{j}}(z, \eta)\right)$ is positive definite, where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor, with $L := F^2$ the complex Lagrangian associated to the complex Finsler function F.

The third condition assures that L is homogeneous with respect to the complex norm, $L(z, \lambda\eta) = \lambda\bar{\lambda}L(z, \eta),\ \forall \lambda \in \mathbb{C}$, and by applying Euler's formula we get that:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i\bar{\eta}^j. \tag{1}$$

An immediate consequence concerns the Cartan complex tensors $C_{i\bar{j}k} := \frac{\partial g_{i\bar{j}}}{\partial \eta^k}$ and $C_{i\bar{j}\bar{k}} := \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}$, which have the following properties:

$$C_{i\bar{j}k} = C_{k\bar{j}i} \quad ; \quad C_{i\bar{j}\bar{k}} = C_{i\bar{k}\bar{j}} \quad ; \quad C_{i\bar{j}k} = \overline{C_{j\bar{i}\bar{k}}} \quad \text{and} \tag{2}$$

$$C_{i\bar{j}k}\eta^k = C_{i\bar{j}\bar{k}}\bar{\eta}^j = C_{i\bar{j}k}\eta^i = C_{i\bar{j}\bar{k}}\bar{\eta}^k = 0 \tag{3}$$

The geometry of complex Finsler spaces consists of the study of geometric objects on the complex manifold $T'M$, endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$. Firstly, we analyse the sections of its complexified tangent bundle $T_{\mathrm{C}}(T'M) = T'(T'M) \oplus T''(T'M)$, with $T''_u(T'M) = \overline{T'_u(T'M)}$. Let $V(T'M) = span\{\frac{\partial}{\partial \eta^k}\} \subset T'(T'M)$ be the vertical bundle and we introduce the complex non-linear connection, denoted by (c.n.c.), as the supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ represent the coefficients of a (c.n.c.). Thus, the pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ represents the adapted frame of the (c.n.c.), which has the dual adapted base $\{\mathrm{d}z^k, \delta\eta^k := \mathrm{d}\eta^k + N_j^k \mathrm{d}z^j\}$.

Further we will use the following notation $\bar{\eta}^j =: \eta^{\bar{j}}$ to denote a conjugate object.

The *CR structure* attempts to describe intrinsically the property of being a hypersurface in complex space; thus, a *CR manifold* can be considered as an embedded CR manifold (hypersurface and edges of wedges in complex space) or as an abstract CR manifold. A *CR-submanifold* $\tilde{M}$ of a Finsler space, extended by S. Dragomir [8, 9], is a real submanifold endowed with a pair of complementary Finslerian distributions $\mathcal{D}$, $\mathcal{D}^\perp \subset T\tilde{M}$, such that $\mathcal{D}$ is invariant, $J(\mathcal{D}_u) = \mathcal{D}_u$, and $\mathcal{D}^\perp$ is anti-invariant, $J(\mathcal{D}_u^\perp) \subset (T_u\tilde{M})^\perp$, for each $u \in \tilde{M}$, where J is an almost complex structure on $\tilde{M}$.

Any real hypersurface $\tilde{M}$ of M is a CR-submanifold, with $\mathcal{D}_u^\perp = J(T_u\tilde{M})^\perp$ and $\mathcal{D}$ the complementary orthogonal distribution of $\mathcal{D}^\perp$.

2 Contact structures subordonated to the CR structure of the complex indicatrix

For (M,F) a complex Finsler manifold, let us take T'_zM its corresponding holomorphic tangent space and F_z the Finsler metric in an arbitrary fixed point

$z \in M$. Then, (T'_zM, F_z) can be regarded as a locally complex n-dimensional Minkowski space, with (η^i) its complex coordinate system, $\eta = (\eta^i) = \eta^i \frac{\partial}{\partial z^i}|_z$. Let g be the Hermitian structure on $T'(\widetilde{T'M})$ associated to F_z, i.e. $g_\eta(Z, W) = g_{j\bar{k}}(z, \eta) Z^j \overline{W^k}$ for any $\eta \in T'_zM$, $Z = Z^j \frac{\partial}{\partial \eta^j}|_\eta$, $W = W^k \frac{\partial}{\partial \eta^k}|_\eta \in T'_\eta(\widetilde{T'_zM})$. As usual in Hermitian geometry we extend g to a complex bilinear form $\mathcal{G}$ on $\widetilde{T'_zM}$ by $\mathcal{G}(Z, \bar{W}) = g(Z, W)$, $\mathcal{G}(Z, W) = \mathcal{G}(\bar{Z}, \bar{W}) = 0$, $\mathcal{G}(\bar{Z}, W) = \overline{\mathcal{G}(Z, \bar{W})}$, $\forall Z, W \in T'(T'_zM)$ which defines a Hermitian metric on $\widetilde{T'_zM}$ and is locally given by:

$$\mathcal{G} := \frac{\partial^2 F_z^2}{\partial \eta^i \partial \bar{\eta}^j} \mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k = g_{j\bar{k}}(z, \eta) \mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k. \tag{4}$$

Any linear connection on M can be extended by linearity to T_CM [12], which is isomorphic to $V_C(T'M)$ via vertical lift. We require ∇ to be a compatible complex connection with respect to the natural complex structure J

$$J(\dot{\partial}_k) = \mathrm{i}\dot{\partial}_k,\ J(\dot{\partial}_{\bar{k}}) = -\mathrm{i}\dot{\partial}_{\bar{k}}, \qquad \text{with} \quad \mathrm{i} := \sqrt{-1}. \tag{5}$$

We can choose ∇ to be the Levi-Civita connection, which is a metrical and symmetric connection and using (2) we get the following components:

$$\Gamma^i_{jk} = g^{\bar{h}i} C_{j\bar{h}k} =: C^i_{jk}(\eta); \quad \Gamma^{\bar{i}}_{\bar{j}k} = 0; \quad \Gamma^i_{\bar{j}k} = 0; \quad \Gamma^{\bar{i}}_{jk} = 0. \tag{6}$$

Since $\Gamma^i_{\bar{j}k} = \Gamma^{\bar{i}}_{\bar{j}k} = 0$, it takes that the Levi-Civita connection is Hermitian, with $C^i_{jk} = C^i_{kj}$ and $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$. Taking into consideration that this Levi-Civita connection is equivalent to the linear Chern connection on $\pi^*T'M = span\{\frac{\partial}{\partial z^i}\}$ [2], where $\pi : T'M \to M$ is the natural projection, and since $C^i_{jk} - C^i_{kj} = 0$, we get that $(\widetilde{T'_zM}, F_z)$ is Kählerian and thus ∇ is Kählerian connection.

For an arbitrary fixed point $z \in M$, the unit sphere in (T'_zM, F_z), also called the *complex indicatrix* in z is:

$$I_zM = \{\eta \in T'_zM \mid F(z, \eta) = 1\}.$$

The positivity of the fundamental tensor $g_{i\bar{j}}$ assures the convexity of the Lagrangian L and the strongly pseudoconvex property of the complex indicatrix I_zM. Moreover, since I_zM has only one defining equation which involves the real valued Finsler function F, it is a real hypersurface of the holomorphic tangent bundle T'_zM, and thus a CR-hypersurface, for any $z \in M$.

Let $(u^1, ..., u^{2n-1})$ be local coordinates on I_zM and $\eta^j = \eta^j(u^1, .., u^{2n-1})$, $\forall j \in \{1, .., n\}$ the equations of inclusion $I_zM \overset{i}{\hookrightarrow} \widetilde{T'_zM}$ [9]. Let us take $l^j = \frac{1}{F}\eta^j$ and $l_j = g_{j\bar{k}} l^{\bar{k}}$, which is equivalent to $l_j = \frac{1}{F}\frac{\partial L}{\partial \eta^j}$ or $l_j = 2\frac{\partial F}{\partial \eta^j}$. If we differentiate with respect to u variable the complex indicatrix condition $F(z, \eta(u)) = 1$, we get

$$l_j \frac{\partial \eta^j}{\partial u^\alpha} + l_{\bar{j}} \frac{\partial \eta^{\bar{j}}}{\partial u^\alpha} = 0, \quad \alpha \in \{1, \ldots, 2n-1\},\ j \in \{1, \ldots, n\}. \tag{7}$$

The tangent map $i_* : T_R(I_z M) \to T_C(\widetilde{T'_z M})$ acts on tangent vectors of $I_z M$ as $i_*\left(\frac{\partial}{\partial u^\alpha}\right) = X_\alpha := \frac{\partial \eta^k}{\partial u^\alpha}\frac{\partial}{\partial \eta^k} + \frac{\partial \bar{\eta}^k}{\partial u^\alpha}\frac{\partial}{\partial \bar{\eta}^k}$. Considering this and (7), we set

$$N = l^j \dot{\partial}_j + l^{\bar{j}} \dot{\partial}_{\bar{j}} \tag{8}$$

and thus we obtain $G_R(X_\alpha, N) = 0$, where G_R is the Riemannian metric applied to real vector fields as

$$G_R(X, Y) = \operatorname{Re} \mathcal{G}(X', \overline{Y'}). \tag{9}$$

with X', $\overline{Y'}$ are the holomorphic, respectively, the anti-holomorphic part of tangent vectors X and Y. Consequently, $N \in T_R(I_z M)^\perp$ and $G_R(N, N) = 1$, so that N is the unit normal vector of the indicatrix bundle.

Since $I_z M$ is a CR-hypersurface with $J\mathcal{D}^\perp = span\{N\}$, we take the characteristic direction of the complex indicatrix CR structure as

$$\xi = JN = \mathrm{i}\left(l^k \dot{\partial}_k - l^{\bar{k}} \dot{\partial}_{\bar{k}}\right), \quad \mathrm{i} := \sqrt{-1}, \tag{10}$$

which is a real tangent unit vector on $I_z M$, with $\xi = \bar{\xi}$, $N = -J\xi$, $\mathcal{D}^\perp = span\{\xi\}$ and $G_R(\xi, \xi) = 1$. Let then $\mathcal{D}$ be the maximal J-invariant subspace of the tangent space of $I_z M$, also called the Levi distribution, orthogonal to $\mathcal{D}^\perp$, such that

$$T_R(I_z M) = \mathcal{D} \oplus span\{\xi\}. \tag{11}$$

Thus, $\dim_R I_z M = 2n - 1$ and $\dim_R \mathcal{D} = 2n - 2$, since $\dim_C M = n$.

Considering (11) and $T_R(T'_z M) = T_R(I_z M) \oplus span\{N\}$ and $T_R(I_z M) = \mathcal{D} \oplus span\{\xi\}$, we can take $\mathcal{D} = T_R(\tilde{M})$, where $\tilde{M}$ is a complex hypersurface of $T'_z M$, with $\dim_C \tilde{M} = n - 1$ and complex unit normal vector $N' = l^j \dot{\partial}_j$. Thus, $\mathcal{D} = \operatorname{Re}\{T'\tilde{M} \oplus T''\tilde{M}\}$ and since $T'(T'_z M) = span\{\dot{\partial}_j\}$, there exist the complex projection factors P^i_a such that

$$T'\tilde{M} = span\{Y'_a := P^j_a \dot{\partial}_j\}, \quad a \in \{1, \ldots, n-1\}.$$

Further, we denote by $\mathcal{D}' := T'\tilde{M}$, $\mathcal{D}'' := T''\tilde{M}$, and so $\mathcal{D} \otimes \mathbb{C} = \mathcal{D}' \oplus \mathcal{D}''$.

Since $Y_a := Y'_a + \overline{Y'_a}$ and $JY_a = \mathrm{i}(Y'_a - \overline{Y'_a})$, we conclude that

$$\mathcal{D} = span\{Y_a := P^j_a \dot{\partial}_j + P^{\bar{j}}_{\bar{a}} \dot{\partial}_{\bar{j}},\ JY_a = \mathrm{i}(P^j_a \dot{\partial}_j - P^{\bar{j}}_{\bar{a}} \dot{\partial}_{\bar{j}})\}, \tag{12}$$

and, from the orthogonality condition between Y'_a and N', with respect to the Hermitian metric $\mathcal{G}$ (4) we have

$$P^j_a l_j = 0, \qquad P^{\bar{j}}_{\bar{a}} l_{\bar{j}} = 0 \qquad \text{and} \qquad l^j l_j = 1. \tag{13}$$

In order to introduce an almost contact structure on the complex indicatrix I_zM, $\dim_R I_zM = 2n-1$, we can choose as Reeb vector the characteristic direction ξ and define the 1-form $\theta(X) = G_R(X,\xi)$ for any $X \in \Gamma(T_R(I_zM))$, more precisely

$$\theta = \frac{\mathrm{i}}{2}(l_{\bar{k}} d\bar{\eta}^k - l_k d\eta^k).$$

It verifies $\theta(\xi) = 1$ and $\theta(X) = 0$, $\forall X \in \Gamma(\mathcal{D})$, so $\ker\theta = \mathcal{D}$, i.e. θ is a pseudohermitian structure on M. Moreover, since I_zM is a pseudoconvex CR manifold, any 1-form θ having this properties is a contact form, such that $\theta \wedge (d\theta)^{n-1} \neq 0$.

By considering the decomposition $X = PX + \theta(X)\xi$, $\forall X \in \Gamma(T_R(I_zM))$, with $PX \in \mathcal{D}$, we define the (1,1) tensor field ϕ as

$$\phi X = J(PX) = JX + \theta(X)N, \qquad \forall\, X \in \Gamma(T_R(I_zM)). \tag{14}$$

Notice that $\phi X = JX$ for $X \in \Gamma(\mathcal{D})$, $\phi^2 X = -X + \theta(X)\xi$, $\phi\xi = 0$ and $\theta(\phi X) = 0$. Therefore, we can state

Proposition 2.1. *On the complex indicatrix I_zM of a complex Finsler space it exists a contact structure associated to the CR structure $(\mathcal{D}, J)$, determined by*

$$\phi = J + \theta \otimes N, \qquad \xi = \mathrm{i}(l^k \dot{\partial}_k - l^{\bar{k}} \dot{\partial}_{\bar{k}}), \qquad \theta = \frac{\mathrm{i}}{2}(l_{\bar{k}} d\bar{\eta}^k - l_k d\eta^k), \tag{15}$$

which is called the natural contact structure *of the complex indicatrix I_zM.*

Moreover, the natural contact structure (15) is subordonated to the CR structure $(I_zM, \mathcal{D})$, i.e. it satisfies $[\xi, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$, equivalent to $\iota_\xi d\theta = 0$ or $\mathcal{L}_\xi \theta = 0$.

Considering the integrability of the CR-structure distributions $\mathcal{D}'$, $\mathcal{D}''$, $\mathcal{D}^\perp$ from

Theorem 2.2. *[13] Let (M, F) be a complex Finsler manifold, $z \in M$ an arbitrary fixed point and I_zM the complex indicatrix. Then the following affirmations hold:*

(a) *the anti-invariant distribution $\mathcal{D}^\perp$ is integrable;*

(b) *even though the complex CR-structures $\mathcal{D}'$, $\mathcal{D}''$ of $\mathcal{D}$, $\mathcal{D} \otimes \mathbb{C} = \mathcal{D}' \oplus \mathcal{D}''$, are integrable, the real invariant distribution $\mathcal{D}$ is no involutive, nor integrable.*

and using the following theorem

Theorem 2.3 ([15])**.** *For an almost contact metric manifold the contact structure (ϕ, ξ, θ, g) is normal if and only if D' and $D'' \oplus \langle \xi \rangle^c$ (or D'' and $D' \oplus \langle \xi \rangle^c$) are integrable and $\mathcal{L}_\xi \theta = 0$, where $D = \ker \theta$ and $D \otimes \mathbb{C} = D' \oplus D''$.*

we can state

Proposition 2.4. *Any almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ subordonated to the complex indicatrix CR structure $(I_z M, \mathcal{D})$, in particular the natural one, is normal.*

Therefore, we have

Theorem 2.5. *Let M be a complex Finsler manifold and z an arbitrary fixed point. The complex indicatrix $I_z M$ is a Sasakian manifold.*

3 Classification of indicatrices contact structures

In order to introduce the symmetry classes defined by D. Chinea and C. Gonzales in [6], we firstly recall that the existence of an almost contact metric structure is equivalent to the existence of a reduction of the structural group $O(2n+1)$ to $U(n) \times 1$. Moreover, the covariant derivative $\nabla\Omega$ of the fundamental 2-form $\Omega(X, Y) = g(X, \phi Y)$ of any almost contact manifold is a covariant tensor of degree 3 which has various symmetry proprieties.

Therefore, by taking V, $\dim V = 2n + 1$, a real vector space with an almost contact structure (ϕ, ξ, θ), D. Chinea and C. Gonzales obtained in [6] the decomposition of the vector space of 3-forms on V having the same symmetries of $\nabla\Omega$, namely

$$\begin{aligned} \mathcal{C}(V) = \; & \{\alpha \in \otimes_3^0 V : \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(z, \phi y, \phi z) + \\ & +\theta(y)\alpha(x, \xi, z) + \theta(z)\alpha(x, y, \xi)\} = \bigoplus_{i=1,..,12} \mathcal{C}_i(V), \quad \text{where} \end{aligned}$$

$$
\begin{aligned}
\mathcal{C}_1(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,x,y) = \alpha(x,y,\xi) = 0\}, \\
\mathcal{C}_2(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \sum_{(x,y,z)} \alpha(x,y,z) = 0,\ \alpha(x,y,\xi) = 0\}, \\
\mathcal{C}_3(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) - \alpha(\phi x, \phi y, z) = 0,\ c_{12}\alpha = 0\}, \\
\mathcal{C}_4(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = \tfrac{1}{2(n-1)}[(<x,y> - \theta(x)\theta(y))c_{12}\alpha(z) \\
& \quad -(<x,z> - \theta(x)\theta(z))c_{12}\alpha(y) - <x,\phi y> c_{12}\alpha(\phi z) \\
& \quad + <x,\phi z> c_{12}\alpha(\phi y)],\ c_{12}\alpha(\xi) = 0\}, \\
\mathcal{C}_5(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = \tfrac{1}{2n}[<x,\phi z> \theta(y)\bar{c}_{12}\alpha(\xi) - \\
& \quad - <x,\phi y> \theta(z)\bar{c}_{12}\alpha(\xi)]\}, \\
\mathcal{C}_6(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = \tfrac{1}{2n}[<x,y> \theta(z)c_{12}\alpha(\xi) - \\
& \quad - <x,z> \theta(y)c_{12}\alpha(\xi)]\}, \\
\mathcal{C}_7(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = \theta(z)\alpha(y,x,\xi) - \theta(y)\alpha(\phi x, \phi z, \xi), \\
& \quad c_{12}\alpha(\xi) = 0\}, \\
\mathcal{C}_8(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = -\theta(z)\alpha(y,x,\xi) - \theta(y)\alpha(\phi x, \phi z, \xi), \\
& \quad \bar{c}_{12}\alpha(\xi) = 0\}, \\
\mathcal{C}_9(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = \theta(z)\alpha(y,x,\xi) + \theta(y)\alpha(\phi x, \phi z, \xi)\}, \\
\mathcal{C}_{10}(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = -\theta(z)\alpha(y,x,\xi) + \theta(y)\alpha(\phi x, \phi z, \xi)\}, \\
\mathcal{C}_{11}(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = -\theta(x)\alpha(\xi, \phi y, \phi z)\}, \\
\mathcal{C}_{12}(V) = \quad & \{\alpha \in \mathcal{C}(V): \quad \alpha(x,y,z) = \theta(x)\theta(y)\alpha(\xi,\xi,z) + \theta(x)\theta(z)\alpha(\xi,y,\xi)\},
\end{aligned}
$$

where $c_{12}\alpha(x) = \sum \alpha(e_i, e_i, x)$ and $\bar{c}_{12}\alpha(x) = \sum \alpha(e_i, \phi e_i, x)$, for $x \in V$ and an arbitrary orthonormal base $\{e_i\}$, $i = 1, 2, .., 2n+1$.

The $\mathcal{C}_i(V)$ classes are mutually orthogonal, irreducible and invariant subspaces under action of $\mathcal{U}(n) \times 1$. If we apply this algebraic decomposition to the geometry of almost contact structures, for each invariant subspace we obtain a different class of almost contact metric manifolds. For example, $\mathcal{C}_6$ corresponds to the class of α-Sasakian manifolds, $\mathcal{C}_2 \oplus \mathcal{C}_9$ to the class of almost cosymplectic manifolds and the direct sum $\mathcal{C}_3 \oplus \mathcal{C}_8$ to that one of normal manifolds. More exactly, we will say that a manifold M is of class C_k, $k = 1, .., 12$, if the 3-form $(\nabla\Omega)_x$ belongs to $\mathcal{C}_k(T_xM)$, for any arbitrary $x \in M$.

Using the following results from [6] and [11]

Proposition 3.1. *Any differentiable manifold of real odd dimension, endowed with a strictly pseudoconvex CR structure of hypersurface type and an associated almost contact structure is of* $\mathcal{C}_6 \oplus \mathcal{C}_9$ *class.*

Corollary 3.2. *The manifold* M *is of class* $\mathcal{C}_6$ *if and only if its almost contact structure is normal.*

and the normality of the normal contact metric structure on I_zM, we can state

Theorem 3.3. *The natural contact structure of the indicatrix of a complex Finsler space* $(I_zM, \phi, \xi, \theta)$ *is of class* $\mathcal{C}_6$.

However, the 1-form θ which defines the invariant distribution $\mathcal{D}$ is not unique. So, starting from the natural contact structure (15), we can determine another almost contact structure associated to the same CR-distribution on $(I_zM, \mathcal{D})$.

A special case of transformation between two almost contact manifolds subordonated to the same almost complex distribution $\mathcal{D}$ is the *gauge transformation* of the 1-form θ, given as $\theta \mapsto \tilde{\theta} = \varepsilon e^f \theta$, with $f \in C^\infty(I_zM)$ and $\varepsilon = \pm 1$. It can be easily noticed that 1-forms θ and $\tilde{\theta}$ define the same distribution $\mathcal{D}$. In general, the complex involutivity is invariant under gauge transformations.

Proposition 3.4. *Two almost contact structures* (ϕ, ξ, θ), $(\tilde{\phi}, \tilde{\xi}, \tilde{\theta})$ *are subordonated to the same strict pseudoconvex CR-structure iff it exists a function* $f \in C^\infty(I_zM)$ *such that*

$$\tilde{\theta} = \varepsilon e^f \theta, \qquad \tilde{\xi} = \varepsilon e^{-f}(\xi + \phi A), \qquad \tilde{\phi} = \phi + \theta \otimes A,$$

with $A \in \mathcal{D}$ *defined by* $d\theta(\phi A, X) = df(X) = X(f), \forall X \in \Gamma(\mathcal{D})$.

Applying G_R from (9) to the real tangent vectors from (12) generating $\mathcal{D}$, which have the components on the holomorphic and anti-holomorphic bundles as $Y'_a = P^j_a \dot{\partial}_j$, $\overline{Y'_a} = P^{\bar{j}}_{\bar{a}} \dot{\partial}_{\bar{j}}$, $JY'_a = \mathrm{i} P^i_a \dot{\partial}_i$ and $\overline{JY'_a} = -\mathrm{i} P^{\bar{j}}_{\bar{a}} \dot{\partial}_{\bar{j}}$, by direct calculus we get

$$\begin{aligned} G_R(Y_a, Y_b) = &\quad G_R(JY_a, JY_b) = \mathrm{Re}(g_{j\bar{k}} P^j_a P^{\bar{k}}_{\bar{b}}) =: \mathrm{Re}(g_{a\bar{b}}) \quad \text{and} \\ G_R(Y_a, JY_b) = &\quad -G_R(JY_a, Y_b) = -\mathrm{Re}(\mathrm{i} g_{j\bar{k}} P^j_a P^{\bar{k}}_{\bar{b}}) =: -\mathrm{Re}(\mathrm{i} g_{a\bar{b}}), \end{aligned}$$

with $a, b \in \{1, \dots, n-1\}$. Also, using conditions (13) and $N' = l^j \dot{\partial}_j$, $\overline{N'} = l^{\bar{j}} \dot{\partial}_{\bar{j}}$, $\xi' = \mathrm{i} l^j \dot{\partial}_j$, $\overline{\xi'} = -\mathrm{i} l^{\bar{j}} \dot{\partial}_{\bar{j}}$, with $\mathrm{i} = \sqrt{-1}$, we can easily verify

$$G_R(\xi, Y_a) = G_R(\xi, JY_a) = G_R(N, Y_a) = G_R(N, JY_a) = 0,$$

Therefore, we obtain

$$G_R(JX, JY) = G_R(X, Y) \text{ and } G_R(X, JY) = -G(JX, Y), \ \forall X, Y \in T_R(T'_zM)).$$

Thus, using the above relations, the form of ϕX from (14), $\xi = JN$ and $\theta(X) = G_R(X, \xi)$, we may conclude that, with respect to the natural contact structure (ϕ, ξ, θ) on $(I_zM, \mathcal{D})$, the Riemannian metric G_R satisfies the compatibility conditions

$$G_R(\phi X, \phi Y) = G_R(X, Y) - \theta(X)\theta(Y). \tag{16}$$

Moreover, considering that

$$d\theta = \frac{\mathrm{i}}{F}\left(g_{j\bar{k}} - \frac{1}{2} l_j l_{\bar{k}}\right) d\eta^j \wedge d\bar{\eta}^k,$$

and its action on the real tangent vectors of the complex indicatrix I_zM as

$$\begin{aligned} d\theta(Y_a, Y_b) &= d\theta(JY_a, JY_b) &&= \tfrac{1}{F}\mathrm{Re}(\mathrm{i}\, g_{a\bar{b}}), \\ d\theta(Y_a, JY_b) &= -d\theta(JY_a, Y_b) &&= \tfrac{1}{F}\mathrm{Re}(g_{a\bar{b}}), \\ d\theta(Y_a, \xi) &= d\theta(JY_a, \xi) &&= 0, \end{aligned}$$

we may conclude that $d\theta(X, Y) = \frac{1}{F} G_R(\phi X, Y)$, which are equal for $F = 1$ on I_zM. Hence, we obtain $G_R(X, Y) = d\theta(X, \phi Y) + \theta(X)\theta(Y)$, $\forall X, Y \in T_R(I_zM)$.

Regarding the almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\theta})$ obtained under a gauge transformation, the metric will not fulfill $\tilde{G}(X, Y) = d\tilde{\theta}(X, \tilde{\phi}Y)$ for $X, Y \in \mathcal{D}$. If we require the restrictions of G_R and $\tilde{G}$ to be related by a conformal transformation on $\mathcal{D}$ the new metric will have the expression

$$\begin{aligned} \tilde{G}(X, Y) = \quad & e^{2f}[G_R(X, Y) - \theta(X)G_R(\phi A, Y) - \theta(Y)G_R(\phi A, X) \\ & + G_R(A, A)\theta(X)\theta(Y)], \end{aligned}$$

with $A \in \mathcal{D}$ given by $d\theta(\phi A, X) = df(X) = X(f)$, $X, Y \in \mathcal{D}$. It takes that it verifies $\tilde{G}(X, Y) = e^f d\tilde{\theta}(X, \tilde{\phi}Y)$, $\forall X, Y \in \Gamma(\mathcal{D})$.

Assuming $\varepsilon = 1$, we take $\tilde{\theta} = \frac{\mathrm{i}}{2} e^f (l_{\bar{k}} d\bar{\eta}^k - l_k d\eta^k)$ a gauge transformation and the vector field

$$A = \lambda^a Y_a + \mu^a JY_a, \quad \text{with } \lambda^a, \mu^a \in C^\infty(I_zM),$$

which satisfies $d\theta(\phi A, X) = X(f)$, $X \in \mathcal{D}$, which is equivalent to $\frac{1}{F} G_R(A, X) = -PX(f) \in \mathcal{D}$. Then, the new almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\theta}, \tilde{G})$ on I_zM has

$$\begin{aligned} \tilde{\xi} &= e^{-f}(\xi + \lambda^a JY_a - \mu^a Y_a) \qquad \text{și} \\ \tilde{\phi} &= J + \theta \otimes N + \lambda^a \theta \otimes Y_a + \mu^a \theta \otimes JY_a. \end{aligned}$$

Moreover, the action of the new metric $\tilde{G}$ is

$$\begin{aligned} \tilde{G}(Y_a, Y_b) &= \tilde{G}(JY_a, JY_b) = e^{2f}\mathrm{Re}(g_{a\bar{b}}), & \tilde{G}(Y_a, \xi) &= e^{2f} JY_a(f), \\ \tilde{G}(Y_a, JY_b) &= -\tilde{G}(JY_a, Y_b) = -e^{2f}\mathrm{Re}(\mathrm{i} g_{a\bar{b}}), & \tilde{G}(JY_a, \xi) &= -e^{2f} Y_a(f), \\ \tilde{G}(\xi, \xi) &= e^{2f}(1 + ||A||^2), \qquad \tilde{G}(\tilde{\xi}, \tilde{\xi}) = 1, & \tilde{G}(\xi, \tilde{\xi}) &= e^f, \end{aligned}$$

where $||A||^2 = G_R(A,A) = \lambda^a Y_a(f) + \mu^a JY_a(f)$.

Let us take $\tilde{\nabla}$ and $\tilde{\Omega}$ the Levi-Civita connection and the fundamental 2-form of $(\tilde{\phi},\tilde{\xi},\tilde{\theta},\tilde{G})$, respectively. Then we deduce

$$\tilde{\Omega}(X,Y) = \tilde{G}(\tilde{\phi}X,Y) = e^{2f}[\Omega(X,Y) + F\theta(Y)PX(f) - F\theta(X)PY(f)], \quad (17)$$

where $\Omega(X,Y) = G_R(\phi X,Y) = Fd\theta(X,Y)$, $X,Y \in \chi(I_zM)$.

The normality of (ϕ,ξ,θ,G_R) and $(\tilde{\phi},\tilde{\xi},\tilde{\theta},\tilde{G})$ structures, deduced from Proposition 2.4, is equivalent to $(\mathcal{L}_\xi\phi)(X) = (\tilde{\mathcal{L}}_{\tilde{\xi}}\tilde{\phi})(X) = 0$, $\forall X \in T(I_zM)$. Using

$$\begin{aligned}(\tilde{\mathcal{L}}_{\tilde{\xi}}\tilde{\phi})(X) = \quad & e^{-f}\{(\mathcal{L}_\xi\phi)(X) + (\phi X(f) + \theta(X)A(f))(\xi+\phi A)\\ & +[\phi A,\phi X] - \phi[\phi A,X] + X(f)A + \theta(X)[\xi+\phi A,A]\}\end{aligned}$$

we obtain the following normality condition for A

$$[\phi A,\phi X] - \phi[\phi A,X] = -\phi X(f)(\xi+\phi A) - X(f)A, \quad \forall X \in \Gamma(\mathcal{D}). \quad (18)$$

Using the following relations between $\tilde{\nabla}\tilde{\Omega}$ and $\nabla\Omega$, which can be found in [11],

$$\begin{aligned}(\tilde{\nabla}_X\tilde{\Omega})(Y,Z) = \quad & e^{2f}(\nabla_X\Omega)(Y,Z) + \tfrac{e^{2f}}{2}\{Z(f)G_R(X,\phi Y) - Y(f)G_R(X,\phi Z)\\ & +\phi Z(f)G_R(X,Y) - \phi Y(f)G_R(X,Z)\},\\ (\tilde{\nabla}_X\tilde{\Omega})(\tilde{\xi},Z) = \quad & e^{f}(\nabla_X\Omega)(\xi,Z) - \tfrac{e^{f}}{2}\{\xi(f)G_R(X,\phi Z) - F\phi Z(f)\phi X(f)\\ & -G_R([\phi A,\phi Z] - \phi[\phi A,Z],X) + FZ(f)X(f)\},\\ \tilde{\nabla}_{\tilde{\xi}}\tilde{\Omega} = \quad & 0,\end{aligned}$$

and considering (18) and the following nonzero components of $\nabla\Omega$

$$\begin{aligned}(\nabla_{Y_a}\Omega)(Y_b,\xi) \quad &= -(\nabla_{Y_a}\Omega)(\xi,Y_b) = -\tfrac{1}{F}\mathrm{Re}(g_{b\bar{a}}),\\ (\nabla_{Y_a}\Omega)(JY_b,\xi) \quad &= -(\nabla_{Y_a}\Omega)(\xi,JY_b) = -\tfrac{1}{F}\mathrm{Re}\,(\mathrm{i}g_{b\bar{a}}),\\ (\nabla_{JY_a}\Omega)(Y_b,\xi) \quad &= -(\nabla_{JY_a}\Omega)(\xi,Y_b) = \tfrac{1}{F}\mathrm{Re}(g_{b\bar{a}}),\\ (\nabla_{JY_a}\Omega)(JY_b,\xi) \quad &= -(\nabla_{JY_a}\Omega)(\xi,JY_b) = \tfrac{1}{F}\mathrm{Re}\,(\mathrm{i}g_{b\bar{a}}),\end{aligned}$$

we obtain the following covariant derivative $\tilde{\nabla}\tilde{\Omega}$ of $\tilde{\Omega}$ as

$$\begin{aligned}
(\tilde{\nabla}_{Y_a}\tilde{\Omega})(Y_b, Y_c) = \quad & -(\tilde{\nabla}_{Y_a}\tilde{\Omega})(JY_b, JY_c) = (\tilde{\nabla}_{JY_a}\tilde{\Omega})(JY_b, Y_c) = \\
& -Fe^{2f}\mathrm{Re}\{\mathrm{i}[P_b^j\dot{\partial}_j(f)g_{c\bar{a}} - P_c^k\dot{\partial}_k(f)g_{b\bar{a}}]\}, \\
(\tilde{\nabla}_{Y_a}\tilde{\Omega})(JY_b, Y_c) = \quad & -(\tilde{\nabla}_{JY_a}\tilde{\Omega})(Y_b, Y_c) = (\tilde{\nabla}_{JY_a}\tilde{\Omega})(JY_b, JY_c) = \\
& Fe^{2f}\mathrm{Re}[P_b^j\dot{\partial}_j(f)g_{c\bar{a}} - P_c^k\dot{\partial}_k(f)g_{b\bar{a}}], \\
(\tilde{\nabla}_{Y_a}\tilde{\Omega})(\tilde{\xi}, Y_b) = \quad & \tfrac{e^f}{F}\mathrm{Re}(g_{b\bar{a}}) + \tfrac{e^f}{2}\xi(f)\mathrm{Re}(\mathrm{i}g_{a\bar{b}}), \\
(\tilde{\nabla}_{Y_a}\tilde{\Omega})(\tilde{\xi}, JY_b) = \quad & \tfrac{e^f}{F}\mathrm{Re}(\mathrm{i}g_{b\bar{a}}) + \tfrac{e^f}{2}\xi(f)\mathrm{Re}(g_{a\bar{b}}), \\
(\tilde{\nabla}_{JY_a}\tilde{\Omega})(\tilde{\xi}, Y_b) = \quad & -\tfrac{e^f}{F}\mathrm{Re}(g_{b\bar{a}}) - \tfrac{e^f}{2}\xi(f)\mathrm{Re}(g_{a\bar{b}}), \\
(\tilde{\nabla}_{JY_a}\tilde{\Omega})(\tilde{\xi}, JY_b) = \quad & \tfrac{e^f}{F}\mathrm{Re}(\mathrm{i}g_{b\bar{a}}) + \tfrac{e^f}{2}\xi(f)\mathrm{Re}(\mathrm{i}g_{a\bar{b}}),
\end{aligned} \tag{19}$$

and, from the general case, $\tilde{\nabla}_{\tilde{\xi}}\tilde{\Omega} = 0$.

By adapting the results from [11] we obtain the classes of a gauge transformation from the normal contact structure of I_zM

Proposition 3.5. *Let us consider I_zM the complex indicatrix of a complex Finsler space M, $\dim_C M = n$, $n \geq 3$. The almost contact metric manifold $(I_zM, \tilde{\phi}, \tilde{\xi}, \tilde{\theta}, \tilde{G})$ obtained under a gauge transformation of the natural contact structure belongs to the $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$ class.*

Each component of $\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$ corresponding to $(I_zM, \tilde{\phi}, \tilde{\xi}, \tilde{\theta}, \tilde{G})$ can be found explicitly using relations (19).

4 Conclusion

Based on the CR-structure of the complex indicatrix I_zM of a complex Finsler space, which was introduced in [13], in the present paper we continue the study by considering contact structures subordonated to the CR-structure of the complex indicatrix. Therefore, the characteristic direction and the maximal J-invariant subspace of the tangent space of I_zM, which characterize the CR-structure, allow us to introduce in this paper a natural contact structure specific to the complex indicatrix. However, our main goal is o characterize the complex indicatrix from an algebraic point of view, using the symmetry classes of the covariant derivative of the fundamental 2-form Ω. In this sense, we use the integrability result from Theorem 2.2, proved in [13], which helps us now to deduce that the complex indicatrix is a Sasakian manifold. In this way, we can make a connection with the general results obtained in [6, 11] and we can state the classes corresponding to the natural contact structure on I_zM. Our next step here is to analyse the properties for the gauge transformations under which it is possible to obtain different types of almost contact metric structures associated with the same CR-structure of the complex indicatrix. As we can see from relations (19) it is quite difficult to obtain directly the algebraic classes of symmetries. Nevertheless, by adapting the results from [11] we obtain the classes of the gauge transformation of the normal contact structure on I_zM and each of their components can be found using (19). Thus, in this paper we make a link between the geometric and the algebraic properties which characterizes the subordonated contact structure of the complex indicatrix CR-structure.

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